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STRESS INTENSITY FACTORS FOR ANNULAR CRACKS IN INHOMOGENEOUS ISOTROPIC MATERIALS

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Abstract—A basic procedure is given for converting a pair of coupled two-dimensional integral equations relevant to mixed boundary value problems with annular type regions to two non-coupled integral equations. The procedure utilizes fundamental solutions to the corresponding internal and external problems in order to separate the coupled equations and write their solution in series form. The method is first applied to axisymmetric shear loading of an annular crack in a homogeneous transversely isotropic material (where the crack plane is parallel to the plane of isotropy) and to uniform tensile loading of an annular crack in a homogeneous isotropic material. Stress intensity factors are compared with previous analysis to test the accuracy of the solution. New results for stress intensity factors are given for the arbitrary normal loading of an annular crack in an inhomogeneous isotropic medium.

1. INTRODUCTION

Many problems in solid mechanics, fluid mechanics, electromagnetics, etc., are all mathematically described by identical equations and fall into the category of potential theory. After the unified governing equations have been established and their general solutions found, a large class of problems can be solved through the same approach. This not only simplifies the solution procedures for various problems but also helps in understanding the relations between different physical phenomena. This paper develops a simple solution procedure appropriate to a class of annular type planar mixed boundary value problems in potential theory. Problems of this nature can be described by a pair of coupled two-dimensional integral equations. Here a simple method utilizing the fundamental solutions for the individual interior and exterior problems is developed to uncouple the equations and write the solution in series form.

Before the method can be applied to various problems, Green's functions for the individual interior and exterior regions must be known. Although the method used here applies to annular type regions with non-circular boundaries and finite bodies, attention will be focused on circular boundaries and infinite bodies since the Green's functions for the interior and exterior domains have closed form analytical expressions. Hence this paper applies the method to analyze the circular annular crack in an infinite medium.

The interior and exterior fundamental solutions for circular cracks have generally been obtained separately, applying the same mathematical method in each case [e.g. Sneddon and Lowengrub (1969); Cherepanov (1979); Kassir and Sih (1975); Guidera and Lardner (1975); Stallybrass (1981); Fabrikant (1989)]. In many circumstances, the two solutions can be directly related through elementary algebraic manipulations. This is illustrated for a circular crack under symmetric tensile concentrated loading in an infinite inhomogeneous isotropic material and for axisymmetric shear loading of an infinite homogeneous transversely isotropic material when the crack plane is parallel to the plane of isotropy. These two cases are chosen since they are needed in subsequent analysis for the annular crack.

The annular crack problem under normal loading in an infinite homogeneous isotropic material has received considerable attention from previous researchers [e.g. Smetanin (1968); Moss and Kobayashi (1971); Mastrojannis and Kermanidis (1981); Choi and Shield (1982); Mastrojannis (1983); Selvadurai and Singh (1985); Clements and Ang (1988)]. The annular crack in a thick walled cylinder under normal loading was analyzed

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by Nied and Erdogan (1983). Axisymmetric torsion of an annular crack in an infinite isotropic material was also studied by Choi and Shield (1982). In this paper the method which is developed is first tested against several of these previous solutions to illustrate the accuracy of the calculations. The method is then used, with no additional difficulty, to analyze the annular crack in an isotropic inhomogeneous material whose elastic modulus varies as a power function of depth under conditions of arbitrary normal loading.

2. FORMULATION

In this section the appropriate integral equations are given and a general form of the solution is obtained. Although subsequent attention will focus on the circular annular crack in an infinite body, the form and solution of the integral equations have a more general application to annular type planar cracks with non-circular boundaries in a finite body. In deriving the integral equations for non-circular annular type cracks, the symbol Ω with attached superscripts and subscripts represents a planar area. The plane containing the crack is denoted by Ω_p which is divided into the four subsets Ω_a, Ω_0^a, Ω_b and Ω_0^b. If $\varnothing$ denotes the empty set, the following relations are assumed to hold between the four subsets as illustrated in Fig. 1 (a–c)

$$\Omega_a \cap \Omega_0^a = \varnothing, \quad \Omega_a \cup \Omega_0^a = \Omega_p, \tag{1}$$

$$\Omega_b \cap \Omega_0^b = \varnothing, \quad \Omega_b \cup \Omega_0^b = \Omega_p, \tag{2}$$

$$\Omega_a \cap \Omega_b = \varnothing, \quad \Omega_0^a \cap \Omega_0^b = \Omega_a^b, \tag{3}$$

$$\Omega_a^b \cup \Omega_b = \Omega_0^a, \quad \Omega_a \cup \Omega_a^b = \Omega_0^b. \tag{4}$$

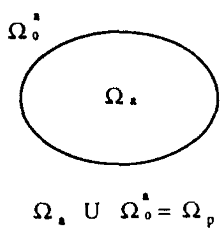

Fig. 1(a). The regions Ω_a and Ω_0^a.

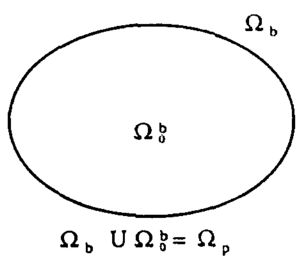

Fig. 1(b). The regions Ω_b and Ω_0^b.

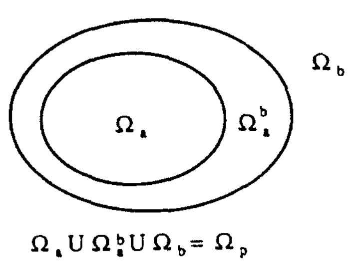

Fig. 1(c). The regions Ω_a, Ω_b and Ω_a^b.

For the special case of a circular geometry to be considered in detail later, Ω_a is a circular area of radius $r = a$, Ω_0^b is a circular area of radius $r = b$, $b > a$, and Ω_a^b is the annular region $a < r < b$.

Deriving the integral equations for the annular type crack utilizes the solution of two fundamental integral equations for the regions Ω_a and Ω_b. These integral equations take the form

$$\int_{\Omega_a} H(\Omega_a, \Omega)\sigma_a^0(\Omega, \Omega_0^a)\,\mathrm{d}\Omega = P_a H(\Omega_a, \Omega_0^a), \tag{5}$$

$$\int_{\Omega_b} H(\Omega_b, \Omega)\sigma_b^0(\Omega, \Omega_0^b)\,\mathrm{d}\Omega = P_b H(\Omega_b, \Omega_0^b), \tag{6}$$

where the kernel H is a known function, P_a and P_b are constants and σ_a^0, σ_b^0 are the fundamental solutions of the integral equations. For compactness the following notation has been employed. Considering a polar coordinate system centered in Ω_a, the notation $H(\Omega_a, \Omega_0^a)$ is equivalent to $H(r, \theta, r_0, \theta_0)$; $r, \theta \in \Omega_a$, $r_0, \theta_0 \in \Omega_0^a$. Physically speaking, in terms of fracture mechanics applications, eqn (5) can be interpreted as an integral equation for the normal stress σ_a^0 in the bond ligament Ω_a of an external crack in an infinite medium loaded by equal and opposite point normal forces P_a at a point on the external crack faces in Ω_0^a. In this interpretation, $H(\Omega_a, \Omega_0^a)$ is the normal displacement on the surface of a half space at a point in Ω_a caused by a unit normal force at a point in Ω_0^a. Equation (6) then corresponds to an internal crack over the area Ω_0^b.

The solutions to the integral equations are assumed to exist in the forms:

$$\sigma_a^0(\Omega_a, \Omega_0^a) = \lambda_a P_a f_a(\Omega_a, \Omega_0^a), \tag{7}$$

$$\sigma_b^0(\Omega_b, \Omega_0^b) = \lambda_b P_b f_b(\Omega_b, \Omega_0^b), \tag{8}$$

where f_a and f_b are known functions in Ω_a and Ω_b respectively, and λ_a and λ_b are two real constants satisfying $\lambda_a < 1$ and $\lambda_b < 1$.

Now consider the following pair of coupled integral equations for the unknown functions σ_a and σ_b:

$$\int_{\Omega_a} H(\Omega_a, \Omega)\sigma_a(\Omega, \Omega_a^b)\,\mathrm{d}\Omega + \int_{\Omega_b} H(\Omega_a, \Omega)\sigma_b(\Omega, \Omega_a^b)\,\mathrm{d}\Omega = PH(\Omega_a, \Omega_a^b), \tag{9}$$

$$\int_{\Omega_a} H(\Omega_b, \Omega)\sigma_a(\Omega, \Omega_a^b)\,\mathrm{d}\Omega + \int_{\Omega_b} H(\Omega_b, \Omega)\sigma_b(\Omega, \Omega_a^b)\,\mathrm{d}\Omega = PH(\Omega_b, \Omega_a^b). \tag{10}$$

If the solutions in eqns (7) and (8) are known, then the coupled equations (9) and (10) can be transformed to the following uncoupled equations:

$$\sigma_a(\Omega_a, \Omega_a^b) = \lambda_a P\left(f_a(\Omega_a, \Omega_a^b) - \lambda_b \int_{\Omega_b} f_a(\Omega_a, \Omega) f_b(\Omega, \Omega_a^b)\,\mathrm{d}\Omega\right) + \lambda_a\lambda_b \int_{\Omega_b} f_a(\Omega_a, \Omega) \int_{\Omega_a} f_b(\Omega, \Omega')\sigma_a(\Omega', \Omega_a^b)\,\mathrm{d}\Omega'\,\mathrm{d}\Omega, \tag{11}$$

$$\sigma_b(\Omega_b, \Omega_a^b) = \lambda_b P\left(f_b(\Omega_b, \Omega_a^b) - \lambda_a \int_{\Omega_a} f_b(\Omega_b, \Omega) f_a(\Omega, \Omega_a^b)\,\mathrm{d}\Omega\right) + \lambda_a\lambda_b \int_{\Omega_a} f_b(\Omega_b, \Omega) \int_{\Omega_b} f_a(\Omega, \Omega')\sigma_b(\Omega', \Omega_a^b)\,\mathrm{d}\Omega'\,\mathrm{d}\Omega. \tag{12}$$

To obtain the uncoupled equations given above, the following procedure may be followed. Temporarily treating σ_b as a known quantity in eqn (9) and moving it to the right hand side, σ_a can be written using eqn (7) as

$$\sigma_a(\Omega_a, \Omega_a^b) = \lambda_a P f_a(\Omega_a, \Omega_a^b) - \lambda_a \int_{\Omega_b} f_a(\Omega_a, \Omega)\sigma_b(\Omega, \Omega_a^b)\, \mathrm{d}\Omega. \tag{13}$$

Similarly, treating σ_a as a known quantity in eqn (10), σ_b can be given by the superposition

$$\sigma_b(\Omega_b, \Omega_a^b) = \lambda_b P f_b(\Omega_b, \Omega_a^b) - \lambda_b \int_{\Omega_a} f_b(\Omega_b, \Omega)\sigma_a(\Omega, \Omega_a^b)\, \mathrm{d}\Omega, \tag{14}$$

where eqn (8) has been used. It is easily verified that eqns (13) and (14) automatically satisfy the coupled eqns (9) and (10). Substituting eqn (14) into eqn (13) leads directly to eqn (11) while substituting eqn (13) into eqn (14) gives eqn (12).

Equations (11) and (12) are Fredholm integral equations of the second kind. General solutions to these equations are given in Neumann series form as:

$$\sigma_a(\Omega_a, \Omega_a^b) = \lambda_a P \sum_{n=0}^{\infty} (-1)^n \lambda_n^a \sigma_{a,n}(\Omega_a, \Omega_a^b), \tag{15}$$

$$\sigma_b(\Omega_b, \Omega_a^b) = \lambda_b P \sum_{n=0}^{\infty} (-1)^n \lambda_n^b \sigma_{b,n}(\Omega_b, \Omega_a^b). \tag{16}$$

The terms in each series are given by the recursion relations

$$\sigma_{a,n}(\Omega_a, \Omega_a^b) = \int_{\Omega_b} f_a(\Omega_a, \Omega)\sigma_{b,n-1}(\Omega, \Omega_a^b)\, \mathrm{d}\Omega, \quad n \geqslant 1, \tag{17}$$

$$\sigma_{b,n}(\Omega_b, \Omega_a^b) = \int_{\Omega_a} f_b(\Omega_b, \Omega)\sigma_{a,n-1}(\Omega, \Omega_a^b)\, \mathrm{d}\Omega, \quad n \geqslant 1, \tag{18}$$

$$\sigma_{a,0}(\Omega_a, \Omega_a^b) = f_a(\Omega_a, \Omega_a^b), \tag{19}$$

$$\sigma_{b,0}(\Omega_b, \Omega_a^b) = f_b(\Omega_b, \Omega_a^b), \tag{20}$$

and

$$\lambda_n^a = \lambda_b \lambda_{n-1}^b, \quad n \geqslant 1, \tag{21}$$

$$\lambda_n^b = \lambda_a \lambda_{n-1}^a, \quad n \geqslant 1, \tag{22}$$

$$\lambda_0^a = \lambda_0^b = 1. \tag{23}$$

The series solution in eqns (15) and (16) is essentially a reflection method or an iterative solution to eqns (13) and (14) [and hence eqns (11) and (12)]. Discussions on convergence of such series solutions can be found in many references on integral equations [see for example Tricomi (1985)] and will not be examined here. Subsequent numerical calculations verify a fast convergence with relatively high accuracy.

The above derivations can be carried out without any reference to a physical background or application of the coupled integral equations (9) and (10). Therefore the solutions in eqns (15) and (16) are of general mathematical significance for various types of problems in potential theory. This solution procedure is certainly not new to potential theory or fracture mechanics, however the applications considered presently do not seem to have been previously analyzed.

It is important to note that the above solution is applicable to annular type regions with boundaries of arbitrary shape (not necessarily circular) and hence is a general solution. For the solution of a particular problem, the main requirement is knowledge of the fundamental solutions of eqns (5) and (6). For fracture mechanics applications, these fundamental solutions correspond to point force loading of the crack faces.

3. SOLUTION TRANSFORMATIONS

As mentioned in the previous section, knowledge of the individual fundamental solutions is required in order to solve the coupled problem. These solutions have been given as analytical expressions only for the case of circular geometries in the past and present literature. Although an analytical form to the solutions is not a requirement, present considerations will be limited to a circular geometry.

Knowledge of both (internal and external) fundamental solutions for a circle is necessary. In some cases, such as the circular internal and external crack in an infinite medium, both fundamental solutions have been previously obtained by methods of potential theory [e.g. Sneddon and Lowengrub (1969); Kassir and Sih (1975); Fabrikant (1989)]. In these cases, the investigators have used the same mathematical method to obtain both solutions independently. On the other hand, in some cases only one of the fundamental solutions may be known. Here a simple method is used illustrating the relationship between the fundamental solutions for a circle which can be used to obtain one from knowledge of the other without solving a similar boundary value problem.

As an example, consider an infinite inhomogeneous medium with the elastic modulus $E = E_0|z|^\alpha$ (E_0 = constant, $0 \leqslant \alpha < 1$) containing a circular crack (internal or external) of radius $r = a$ located on the plane $z = 0$. A polar coordinate system (r, θ) is taken at the crack center. The crack is opened by a pair of concentrated normal forces $\pm P$ at the point $(r_0, \theta_0, 0)$ applied to the upper and lower crack faces. Symmetry allows the reduction to a half space problem with no surface shear stresses. The normal surface displacement (apart from a constant multiplier) of an inhomogeneous half space caused by a unit point normal force has been found previously by Rostovtesv and Khranevskaia (1971) as

$$H(r,\theta,r_0,\theta_0) = [r^2+r_0^2-2rr_0\cos(\theta-\theta_0)]^{-(1+\alpha)/2}. \tag{24}$$

The two fundamental integral equations (5) and (6) can be written as:

(a) for a penny-shaped crack of radius b

$$\int_0^{2\pi}\int_b^{L}\frac{\sigma_b^0(\rho,\phi,r_0,\theta_0)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{[r^2+\rho^2-2r\rho\cos(\phi-\theta)]^{(\alpha+1)/2}} = \frac{P}{[r^2+r_0^2-2rr_0\cos(\theta_0-\theta)]^{(\alpha+1)/2}},$$

$$b < r < \infty,\quad 0 \leqslant \theta \leqslant 2\pi,\quad 0 < r_0 < b,\quad 0 \leqslant \theta_0 \leqslant 2\pi; \tag{25}$$

(b) for a circular external crack of radius a

$$\int_0^{2\pi}\int_0^{a}\frac{\sigma_a^0(\rho,\phi,r_0,\theta_0)\rho\,\mathrm{d}\rho\,\mathrm{d}\phi}{[r^2+\rho^2-2r\rho\cos(\phi-\theta)]^{(\alpha+1)/2}} = \frac{P}{[r^2+r_0^2-2rr_0\cos(\theta_0-\theta)]^{(\alpha+1)/2}}$$
$$+\delta+\gamma r\cos\theta+\beta r\sin\theta,$$

$$0 < r < a,\quad 0 \leqslant \theta \leqslant 2\pi,\quad a < r_0 < \infty,\quad 0 \leqslant \theta_0 \leqslant 2\pi. \tag{26}$$

The additional terms δ, γ and β in eqn (26) are constants which are related to rigid body displacements at infinity which may arise in external crack problems [e.g. Stallybrass (1981); Fabrikant *et al.* (1986); Gao and Rice (1987)]. For solving annular type internal cracks, the case $\delta = \gamma = \beta = 0$ is appropriate which implies the vanishing of all displacements at infinity.

The similarity of the two integral equations is apparent and in fact both can be put into the same form. To illustrate this, substitute

$$x = \frac{b}{\rho}, \quad \bar{r} = \frac{b}{r}, \quad \bar{r}_0 = \frac{b}{r_0}, \tag{27}$$

into eqn (25) resulting in

$$\int_0^{2\pi}\int_0^1 \frac{\bar{\sigma}_b^0(x,\phi,\bar{r}_0,\theta_0)\,\mathrm{d}x\,\mathrm{d}\phi}{[\bar{r}^2+x^2-2\bar{r}x\cos(\phi-\theta)]^{(\alpha+1)/2}} = \frac{P}{[\bar{r}^2+\bar{r}_0^2-2\bar{r}\bar{r}_0\cos(\theta_0-\theta)]^{(\alpha+1)/2}}$$

$$0<\bar{r}<1,\quad 0\leqslant\theta\leqslant 2\pi,\quad 1<\bar{r}_0<\infty,\quad 0\leqslant\theta_0\leqslant 2\pi, \tag{28}$$

where

$$\bar{\sigma}_b^0(x,\phi,\bar{r}_0,\theta_0) = \sigma_b^0\left(\frac{b}{x},\phi,\frac{b}{\bar{r}_0},\theta_0\right)x^{\alpha-2}\bar{r}_0^{-(\alpha+1)}b^2. \tag{29}$$

Similarly substituting

$$x = \frac{\rho}{a}, \quad \bar{r} = \frac{r}{a}, \quad \bar{r}_0 = \frac{r_0}{a}, \tag{30}$$

into eqn (26) gives

$$\int_0^{2\pi}\int_0^1 \frac{\sigma_a^0(ax,\phi,a\bar{r}_0,\theta_0)a^2x\,\mathrm{d}x\,\mathrm{d}\phi}{[\bar{r}^2+x^2-2\bar{r}x\cos(\phi-\theta)]^{(\alpha+1)/2}} = \frac{P}{[\bar{r}^2+\bar{r}_0^2-2\bar{r}\bar{r}_0\cos(\theta_0-\theta)]^{(\alpha+1)/2}}$$

$$0<\bar{r}<1,\quad 0\leqslant\theta\leqslant 2\pi,\quad 1<\bar{r}_0<\infty,\quad 0\leqslant\theta_0\leqslant 2\pi. \tag{31}$$

Equations (28) and (31) have identical forms and hence identical solutions. Having one solution allows the other to be automatically obtained by the relation

$$\sigma_a^0(\rho,\phi,r_0,\theta_0) = \sigma_b^0\left(\frac{ab}{\rho},\phi,\frac{ab}{r_0},\theta_0\right)a^2b^2r_0^{-(1+\alpha)}\rho^{-3+\alpha}. \tag{32}$$

The solutions for both cases have been given recently by Fabrikant (1989) as:

$$\sigma_a^0(\rho,\phi,r_0,\theta_0) = \frac{P}{\pi^2}\cos\frac{\pi\alpha}{2}\left(\frac{r_0^2-a^2}{a^2-\rho^2}\right)^{(1-\alpha)/2}\frac{1}{\rho^2+r_0^2-2\rho r_0\cos(\theta_0-\theta)},$$

$$0<\rho<a,\quad a<r_0<\infty, \tag{33}$$

$$\sigma_b^0(\rho,\phi,r_0,\theta_0) = \frac{P}{\pi^2}\cos\frac{\pi\alpha}{2}\left(\frac{b^2-r_0^2}{\rho^2-b^2}\right)^{(1-\alpha)/2}\frac{1}{\rho^2+r_0^2-2\rho r_0\cos(\theta_0-\theta)},$$

$$b<\rho<\infty,\quad 0<r_0<b. \tag{34}$$

It is easy to verify that eqns (33) and (34) satisfy the relation (32). In the case of a homogeneous material, $\alpha = 0$ and the relation also holds. The same is true for a circular crack in a transversely isotropic material if the crack plane is parallel to the plane of isotropy since in this case the normal stress on the crack plane is identical to the isotropic result (Kassir and Sih, 1975; Fabrikant, 1989). It is important to note that eqn (32) holds only in the case of vanishing displacements and rotations at infinity for the external crack solution. Vanishing of the net force or moment will introduce additional terms in the stress distribution for the external crack.

As a second example consider the case of shear loading. At the present time the fundamental solution for point shear loading of neither an internal nor an external crack in an inhomogeneous material appears to be available. Therefore, attention is directed to an infinite transversely isotropic material where the crack plane is parallel to the plane of isotropy. For this case antisymmetry conditions allow the reduction to an equivalent problem for a half space $z > 0$. Taking a polar coordinate system r, θ on the surface $z = 0$, the tangential surface displacements u_r and u_θ at the point (r, θ) caused by point shear forces T_r and T_θ applied at the point (r_0, θ_0) in the r and θ directions respectively, can be found from Fabrikant (1989) as

$$u_r + \mathrm{i}u_\theta = \frac{1}{2}G_1\frac{1}{R}\left((T_r + \mathrm{i}T_\theta)\,\mathrm{e}^{-\mathrm{i}(\theta-\theta_0)} + \frac{G_2}{G_1}(T_r - \mathrm{i}T_\theta)\frac{r - r_0\,\mathrm{e}^{-\mathrm{i}(\theta-\theta_0)}}{r\,\mathrm{e}^{-\mathrm{i}(\theta-\theta_0)} - r_0}\right), \tag{35}$$

where i is the complex number, $R^2 = r^2 + r_0^2 - 2rr_0\cos(\theta - \theta_0)$ and G_1 and G_2 are material constants. The internal fundamental solutions for point shear loading inside a penny-shaped crack can also be found there. The shear stresses at a point (r, θ) on the crack plane outside the crack are given by

$$\tau_{zr} + \mathrm{i}\tau_{z\theta} = \frac{\sqrt{b^2 - r_0^2}}{\pi^2\sqrt{r^2 - b^2}}\left(\frac{(T_r + \mathrm{i}T_\theta)\,\mathrm{e}^{-\mathrm{i}(\theta-\theta_0)}}{R^2} + \frac{G_2}{G_1}(T_r - \mathrm{i}T_\theta)\,\mathrm{e}^{\mathrm{i}(\theta-\theta_0)}\frac{(3r - r_0\,\mathrm{e}^{\mathrm{i}(\theta-\theta_0)})}{r[r - r_0\,\mathrm{e}^{\mathrm{i}(\theta-\theta_0)}]^2}\right),$$

$$b < r < \infty, \quad 0 < r_0 < b, \tag{36}$$

where the point forces of magnitudes T_r and T_θ, pointing in the radial and tangential directions respectively, are applied at the point (r_0, θ_0) inside the crack of radius b.

Equation (36) displays the coupling between the radial and tangential stress components. The previous solution procedure is developed for one unknown function inside and outide the annular region (as in normal loading) and hence cannot be applied directly to general shear loading. The extension to this case will not be taken up here. Instead axisymmetric shear is considered which eliminates the coupling effect. Integrating eqn (36) gives the axisymmetric shear stresses

$$\tau_{zr}(r, r_0) = \frac{2}{\pi}T_r\frac{\sqrt{b^2 - r_0^2}}{\sqrt{r^2 - b^2}}\frac{r_0^2}{r}\frac{1}{r^2 - r_0^2}, \quad b < r < \infty, \quad 0 < r_0 < b, \tag{37}$$

$$\tau_{z\theta}(r, r_0) = \frac{2}{\pi}T_\theta\frac{\sqrt{b^2 - r_0^2}}{\sqrt{r^2 - b^2}}\frac{r_0^2}{r}\frac{1}{r^2 - r_0^2}, \quad b < r < \infty, \quad 0 < r_0 < b, \tag{38}$$

where T_r and T_θ are now ring loadings pointing in the r and θ directions applied to the penny-shaped crack at a radial distance of r_0. The stress distributions in the above equations are identical and independent of material constants, thus being equal to the isotropic results which has been pointed out previously by Kassir and Sih (1975).

The external solutions for ring loadings of an external circular crack are also needed. These can be obtained by extending results currently available in the literature but here it is illustrated how they can be easily obtained from eqns (37) and (38) with the aid of only algebraic manipulations. The procedure is analogous to the previous example for an inhomogeneous material. The tangential displacements on the surface of a transversely isotropic half space under axisymmetric shear loading can be evaluated by integrating eqn (35) giving

$$u_r(r,r_0) = \frac{1}{2}T_r(G_1+G_2)r_0\int_0^{2\pi}\frac{\cos(\theta-\theta_0)\,d\theta_0}{\sqrt{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)}},\quad 0<r,\ r_0<\infty, \tag{39}$$

$$u_\theta(r,r_0) = \frac{1}{2}T_\theta(G_1-G_2)r_0\int_0^{2\pi}\frac{\cos(\theta-\theta_0)\,d\theta_0}{\sqrt{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)}},\quad 0<r,\ r_0<\infty, \tag{40}$$

where again T_r and T_θ are ring loadings at a radial distance of r_0. The appearance of the material constants only in the form of constant multipliers explains the independence of the surface stresses from the elastic constants. The above displacements can be used to formulate the axisymmetric integral equations for the internal and external cracks, and using the previous transformations they can be put into identical forms. The resulting stresses for an external crack of radius a can be deduced as

$$\tau_{zr}(r,r_0) = \frac{2}{\pi}T_r\frac{\sqrt{r_0^2-a^2}}{\sqrt{a^2-r^2}}\frac{r}{r_0^2-r^2},\quad 0<r<a,\quad a<r_0<\infty, \tag{41}$$

$$\tau_{z\theta}(r,r_0) = \frac{2}{\pi}T_\theta\frac{\sqrt{r_0^2-a^2}}{\sqrt{a^2-r^2}}\frac{r}{r_0^2-r^2},\quad 0<r<a,\quad a<r_0<\infty. \tag{42}$$

It is important to point out that the above procedure is not necessarily new but it is one that is seldom if ever applied in the literature. It provides a simple means of checking solutions presently available if both the internal and external problems have been solved. If only one is known, it allows the other to be easily obtained.

4. APPLICATIONS

In this section the previous results are used to analyze two different problems. In both cases a circular geometry is used. The region Ω_a is interior to a circle of radius a while Ω_b is exterior to a circle of radius b. The cracked region Ω_a^b is thus the annulus $a<r<b$ (Fig. 2). The annular crack is located in an infinite body since the corresponding fundamental solutions are known for this case.

As a first example axisymmetric shear loading is considered. This solution is somewhat simpler since there is no angular dependence. As noted earlier, for axisymmetric shear of an annular crack in a transversely isotropic material, the solution is identical to the isotropic case when the crack plane is parallel to the plane of isotropy. The solution is also the same for radial and tangential shear and hence no subscripts indicating direction need be used. The axisymmetric version of the integral equations (9) and (10) apply where the function $H(\Omega_a,\Omega_a^b)$ is given in eqn (39) for radial shear and eqn (40) for tangential shear. The

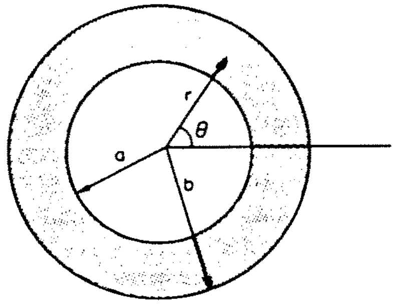

Fig. 2. Annular crack loaded in shear or tension.

axisymmetric form of the solution in Section 2 is given with σ replaced by τ and P by T since the problem is for shear.

The fundamental solutions to the axisymmetric version of eqns (5) and (6) are obtained from the previous section as

$$\tau_a^0(r,r_0) = \frac{2}{\pi}T_a\frac{\sqrt{r_0^2-a^2}}{\sqrt{a^2-r^2}}\frac{r}{r_0^2-r^2}, \quad 0<r<a, \quad a<r_0<\infty, \tag{43}$$

$$\tau_b^0(r,r_0) = \frac{2}{\pi}T_b\frac{\sqrt{b^2-r_0^2}}{\sqrt{r^2-b^2}}\frac{r_0^2}{r}\frac{1}{r^2-r_0^2}, \quad b<r<\infty, \quad 0<r_0<b. \tag{44}$$

From eqns (7) and (8) one may identify $\lambda_a = \lambda_b = 2/\pi$ and

$$f_a(r,r_0) = \frac{\sqrt{r_0^2-a^2}}{\sqrt{a^2-r^2}}\frac{r}{r_0^2-r^2}, \quad 0<r<a, \quad a<r_0<\infty, \tag{45}$$

$$f_b(r,r_0) = \frac{\sqrt{b^2-r_0^2}}{\sqrt{r^2-b^2}}\frac{r_0^2}{r}\frac{1}{r^2-r_0^2}, \quad b<r<\infty, \quad 0<r_0<b. \tag{46}$$

The axisymmetric form of the series solution in eqns (15) and (16) become

$$\tau_a(r,r_0) = \lambda_a T\sum_{n=0}^{\infty}(-1)^n\lambda_n^a\tau_{a,n}(r,r_0), \quad 0<r<a, \quad a<r_0<b, \tag{47}$$

$$\tau_b(r,r_0) = \lambda_b T\sum_{n=0}^{\infty}(-1)^n\lambda_n^b\tau_{b,n}(r,r_0), \quad b<r<\infty, \quad a<r_0<b, \tag{48}$$

with

$$\lambda_n^a = \lambda_n^b = (2/\pi)^n, \quad n\geqslant 0, \tag{49}$$

$$\tau_{a,0}(r,r_0) = \frac{\sqrt{r_0^2-a^2}}{\sqrt{a^2-r^2}}\frac{r}{(r_0^2-r^2)}, \quad 0<r<a, \quad a<r_0<b, \tag{50}$$

$$\tau_{b,0}(r,r_0) = \frac{\sqrt{b^2-r_0^2}}{\sqrt{r^2-b^2}}\frac{r_0^2}{r}\frac{1}{(r^2-r_0^2)}, \quad b<r<\infty, \quad a<r_0<b, \tag{51}$$

$$\tau_{a,n}(r,r_0) = \frac{r}{\sqrt{a^2-r^2}}\int_b^{\infty}\frac{\sqrt{\rho^2-a^2}}{(\rho^2-r^2)}\tau_{b,n-1}(\rho,r_0)\,\mathrm{d}\rho, \quad 0<r<a, \quad a<r_0<b, \quad n\geqslant 1, \tag{52}$$

$$\tau_{b,n}(r,r_0) = \frac{1}{r\sqrt{r^2-b^2}}\int_0^a\frac{\sqrt{b^2-\rho^2}}{(r^2-\rho^2)}\rho^2\tau_{a,n-1}(\rho,r_0)\,\mathrm{d}\rho, \quad b<r<\infty, \quad a<r_0<b, \quad n\geqslant 1. \tag{53}$$

The above solution in eqns (47) and (48) corresponds to a ring loading of magnitude T at a radial distance of r_0 from the center. For a distributed loading $\tau(r)$ on $a<r<b$, one may replace T by $\tau(r_0)\,\mathrm{d}r_0$ and integrate on $a<r_0<b$. The parameter r_0 only appears directly in the initial terms $\tau_{a,0}(r,r_0)$ and $\tau_{b,0}(r,r_0)$. For a distributed loading those may be replaced by

$$\tau_{a,0}(r) = \frac{r}{\sqrt{a^2-r^2}} \int_a^b \frac{\sqrt{r_0^2-a^2}}{(r_0^2-r^2)} \tau(r_0)\,\mathrm{d}r_0, \quad 0<r<a. \tag{54}$$

$$\tau_{b,0}(r) = \frac{1}{r\sqrt{r^2-b^2}} \int_a^b \frac{\sqrt{b^2-r_0^2}}{(r^2-r_0^2)} r_0^2 \tau(r_0)\,\mathrm{d}r_0, \quad b<r<\infty, \tag{55}$$

and the argument r_0 in the terms $\tau_a(r, r_0)$, $\tau_{a,n}(r, r_0)$, $\tau_b(r, r_0)$ and $\tau_{b,n}(r, r_0)$ may be dropped.

As a numerical example linear shear loading is considered since this case has been previously examined and hence allows the accuracy of the present method to be evaluated. The distributed shear loading takes the form $\tau(r) = \tau_0(r/b)$ where τ_0 is a constant. Since the solution applies to both radial and tangential shear, the stress intensity factors K_{II} and K_{III} are equal and are given as

$$\begin{bmatrix} K_{\mathrm{II}}^{a} \\ K_{\mathrm{III}}^{a} \end{bmatrix} = \lim_{r\to a^-} \sqrt{2(a-r)}\,\tau_a(r), \tag{56}$$

$$\begin{bmatrix} K_{\mathrm{II}}^{b} \\ K_{\mathrm{III}}^{b} \end{bmatrix} = \lim_{r\to b^+} \sqrt{2(r-b)}\,\tau_b(r), \tag{57}$$

where $\tau_a(r)$ and $\tau_b(r)$ are given in eqns (47) and (48) respectively. The non-dimensionalized stress intensity factors $\bar{K}_{\mathrm{II}}$ and $\bar{K}_{\mathrm{III}}$ (given as $K = \tau_0\sqrt{b}\bar{K}$) are listed in Table 1 for various a/b ratios. The torsion case was analyzed by Choi and Shield (1982) and the present results appear to be in very good agreement with theirs although they did not give explicit numerical values. The numerical results in Table 1 were obtained with 20 terms in each series which provided sufficient accuracy.

The integrations involved are of elementary functions and in some cases they may be analytically evaluated. As an example consider the above case of linear shear loading. The integrations in eqns (54) and (55) are evaluated in closed form as

$$\tau_{a,0}(r) = \frac{\tau_0 r}{b}\left[\frac{\sqrt{b^2-a^2}}{\sqrt{a^2-r^2}} - \frac{\pi}{4} - \frac{1}{2}\sin^{-1}\left(\frac{b^2+r^2-2a^2}{b^2-r^2}\right)\right], \tag{58}$$

$$\tau_{b,0}(r) = \frac{\tau_0}{b}\left[-\frac{(b^2-a^2)^{3/2}}{3r\sqrt{r^2-b^2}} + r\left\{\frac{\sqrt{b^2-a^2}}{\sqrt{r^2-b^2}} - \frac{\pi}{4} - \frac{1}{2}\sin^{-1}\left(\frac{2b^2-a^2-r^2}{r^2-a^2}\right)\right\}\right]. \tag{59}$$

Inserting these expressions into eqns (52) and (53) with $n = 1$ gives the terms $\tau_{a,1}(r)$ and $\tau_{b,1}(r)$ in integral form. These integrals are not evaluated here for arbitrary r but for $r = a$ and $r = b$ they can be easily evaluated. This leads to the following two term approximations to the stress intensity factors

Table 1. Non-dimensionalized stress intensity factors for an annular crack in an infinite homogeneous body under linear shear loading

a/b	0.01	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	0.99
$\bar{K}_{\mathrm{II}}^{a}, \bar{K}_{\mathrm{III}}^{a}$	0.054	0.170	0.235	0.279	0.308	0.322	0.323	0.309	0.274	0.209	0.070
$\bar{K}_{\mathrm{II}}^{b}, \bar{K}_{\mathrm{III}}^{b}$	0.424	0.424	0.423	0.418	0.410	0.395	0.373	0.339	0.291	0.215	0.070

$$\begin{bmatrix} K_{\mathrm{II}}^{a} & K_{\mathrm{III}}^{a} \\ K_{\mathrm{II}}^{b} & K_{\mathrm{III}}^{b} \end{bmatrix} = \tau_0\sqrt{b}\begin{bmatrix} \bar{K}_{\mathrm{II}}^{a} & \bar{K}_{\mathrm{III}}^{a} \\ \bar{K}_{\mathrm{II}}^{b} & \bar{K}_{\mathrm{III}}^{b} \end{bmatrix}$$

$$K^a = \tau_0\sqrt{b}\sqrt{\varepsilon}\sqrt{1-\varepsilon^2}\,\frac{2}{\pi}\left(\frac{2}{\pi}+\frac{1}{3\pi\varepsilon}(1-\varepsilon^2)\ln\frac{1+\varepsilon}{1-\varepsilon}\right). \tag{60}$$

$$K^b = \tau_0\sqrt{b}\left(\frac{2}{\pi}\sqrt{1-\varepsilon^2}-\frac{2}{3\pi}(1-\varepsilon^2)^{3/2}-\frac{4}{\pi^2}\left\{-\frac{\pi}{6}+\left(\frac{\pi}{2}-\frac{\varepsilon}{3}\right)\sqrt{1-\varepsilon^2}\right.\right.$$

$$\left.\left.-\frac{\pi}{6}(1-\varepsilon^2)^{3/2}-\frac{1}{6}(1-\varepsilon^2)^{3/2}\ln\frac{1+\varepsilon}{1-\varepsilon}-\frac{1}{3}\sin^{-1}(1-2\varepsilon^2)\right\}\right), \tag{61}$$

where $\varepsilon = a/b$. The above approximate expressions are exact for $\varepsilon \to 0$ since the subsequent interactions vanish in this limit. They provide a very good accuracy out to a value of $\varepsilon = 0.4$ where the above equations give values of 0.305 and 0.407 which are within 1% of the numerical values in Table 1 which are 0.308 and 0.410.

The limit $a/b \to 1$ corresponds to a two-dimensional crack of length $(b-a)$ under uniform in-plane or anti-plane shear of magnitude τ_0. The stress intensity factors for this limit become

$$K^a = K^b = \tau_0\sqrt{(b-a)/2} = \tau_0\sqrt{b}\sqrt{0.5(1-a/b)}, \tag{62}$$

which gives a value of 0.0707 for $a/b = 0.99$ which also agrees very well with the numerical result.

Now the annular crack in an inhomogeneous material under normal loading is analyzed (Fig. 2). The inhomogeneous material $(-\infty < z < \infty)$ has a power law variation in the modulus given as $E = E_0|z|^\alpha$ (E_0 = constant, $0 \leqslant \alpha < 1$) and the annular crack is located on the $z = 0$ plane. The integral equations are again given by eqns (9) and (10) where $H(\Omega_a, \Omega_a^h)$ is given in eqn (24) apart from a constant multiplier. The fundamental solutions are given in eqns (33) and (34). From these one may identify

$$\lambda_a = \lambda_b = \frac{1}{\pi^2}\cos\left(\frac{\pi\alpha}{2}\right), \tag{63}$$

$$f_a(r,\theta,r_0,\theta_0) = \left(\frac{r_0^2-a^2}{a^2-r^2}\right)^{(1-\alpha)/2}\frac{1}{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)},\quad 0<r<a,\quad a<r_0<\infty, \tag{64}$$

$$f_b(r,\theta,r_0,\theta_0) = \left(\frac{b^2-r_0^2}{r^2-b^2}\right)^{(1-\alpha)/2}\frac{1}{r^2+r_0^2-2rr_0\cos(\theta-\theta_0)},\quad b<r<\infty,\quad 0<r_0<b. \tag{65}$$

The series solutions are given in eqns (15) and (16) with

$$\lambda_n^a = \lambda_n^b = \left[\frac{1}{\pi^2}\cos\left(\frac{\pi\alpha}{2}\right)\right]^n,\quad n \geqslant 0, \tag{66}$$

$$\sigma_{a,0}(r,\theta,r_0,\theta_0) = f_a(r,\theta,r_0,\theta_0),\quad 0<r<a,\quad a<r_0<b, \tag{67}$$

$$\sigma_{b,0}(r,\theta,r_0,\theta_0) = f_b(r,\theta,r_0,\theta_0),\quad b<r<\infty,\quad a<r_0<b, \tag{68}$$

$$\sigma_{a,n}(r,\theta,r_0,\theta_0) = (a^2-r^2)^{-(1-\alpha)/2}\int_0^{2\pi}\int_b^{\infty}(\rho^2-a^2)^{(1-\alpha)/2}\frac{\sigma_{b,n-1}(\rho,\phi,r_0,\theta_0)}{r^2+\rho^2-2r\rho\cos(\phi-\theta)}\rho\,d\rho\,d\phi,$$

$$0<r<a,\quad a<r_0<b,\quad n \geqslant 1, \tag{69}$$

$$\sigma_{b,n}(r,\theta,r_0,\theta_0) = (r^2-b^2)^{-(1-\alpha)/2}\int_0^{2\pi}\int_0^{a}(b^2-\rho^2)^{(1-\alpha)/2}\frac{\sigma_{a,n-1}(\rho,\phi,r_0,\theta_0)}{r^2+\rho^2-2r\rho\cos(\phi-\theta)}\rho\,\mathrm{d}\rho\,\mathrm{d}\phi,$$

$$b<r<\infty,\quad a<r_0<b,\quad n\geqslant 1. \tag{70}$$

The above solution is for concentrated loading at the point (r_0, θ_0). A distributed loading on the crack face can be obtained by integration. The pressure analyzed here is of the form

$$p(r,\theta) = \sigma_0 Q(r)\cos(m\theta),\quad a<r<b,\quad 0<\theta<2\pi, \tag{71}$$

where σ_0 is a constant, $Q(r)$ is the axisymmetric part of the load and m is zero or a positive integer. Replacing P by $\sigma_0 Q(r_0)\cos(m\theta_0)r_0\,\mathrm{d}r_0\,\mathrm{d}\theta_0$ and integrating on $a<r_0<b$, $0<\theta_0<2\pi$ gives the solution

$$\sigma_a(r,\theta) = \lambda_a\cos(m\theta)\sum_{n=0}^{\infty}(-1)^n\lambda_n^a\sigma_{a,n}(r),\quad 0<r<a, \tag{72}$$

$$\sigma_b(r,\theta) = \lambda_b\cos(m\theta)\sum_{n=0}^{\infty}(-1)^n\lambda_n^b\sigma_{b,n}(r),\quad b<r<\infty, \tag{73}$$

where λ_a, λ_b, λ_n^a and λ_n^b are as above with $\sigma_{a,n}(r)$ and $\sigma_{b,n}(r)$ now given by

$$\sigma_{a,0}(r) = \frac{2\pi\sigma_0 r^m}{(a^2-r^2)^{(1-\alpha)/2}}\int_a^b\frac{(r_0^2-a^2)^{(1-\alpha)/2}r_0^{1-m}}{r_0^2-r^2}Q(r_0)\,\mathrm{d}r_0,\quad 0<r<a, \tag{74}$$

$$\sigma_{b,0}(r) = \frac{2\pi\sigma_0 r^{-m}}{(r^2-b^2)^{(1-\alpha)/2}}\int_a^b\frac{(b^2-r_0^2)^{(1-\alpha)/2}r_0^{1+m}}{r^2-r_0^2}Q(r_0)\,\mathrm{d}r_0,\quad b<r<\infty, \tag{75}$$

$$\sigma_{a,n}(r) = \frac{2\pi r^m}{(a^2-r^2)^{(1-\alpha)/2}}\int_b^{\infty}\frac{(\rho^2-a^2)^{(1-\alpha)/2}\rho^{1-m}}{\rho^2-r^2}\sigma_{b,n-1}(\rho)\,\mathrm{d}\rho,\quad 0<r<a,\quad n\geqslant 1, \tag{76}$$

$$\sigma_{b,n}(r) = \frac{2\pi r^{-m}}{(r^2-b^2)^{(1-\alpha)/2}}\int_0^a\frac{(b^2-\rho^2)^{(1-\alpha)/2}\rho^{1+m}}{r^2-\rho^2}\sigma_{a,n-1}(\rho)\,\mathrm{d}\rho,\quad b<r<\infty,\quad n\geqslant 1. \tag{77}$$

It is evident from the above solution that the stress field does not exhibit a square root singularity at the crack front as in homogeneous materials. Therefore numerical results are given for the stress intensity factors defined as

$$K_\mathrm{I}^a = \lim_{r\to a^-}[2(a-r)]^{(1-\alpha)/2}\sigma_a(r,\theta), \tag{78}$$

$$K_\mathrm{I}^b = \lim_{r\to b^+}[2(r-b)]^{(1-\alpha)/2}\sigma_b(r,\theta). \tag{79}$$

In the limit $\alpha = 0$, the results for a homogeneous material are obtained and the stress field becomes square root singular at the crack front. Note that for general values of α, the singularity is between zero and one-half, less than for a homogeneous material.

Two different loading conditions were used for the numerical solution. The first case corresponds to uniform internal pressure of magnitude σ_0 and can be obtained by taking $m = 0$ and $Q(r) = 1$. Temporarily concentrating attention on the homogeneous case $\alpha = 0$, the non-dimensionalized stress intensity factors $\bar{K}_\mathrm{I}^a$ and $\bar{K}_\mathrm{I}^b$ are defined as $K_I =$

$\sigma_0 b^{1/2}\bar{K}_{\rm I}$. The first four columns of Table 2 compare the present numerical solution with that of Clements and Ang (1988). They displayed that their results were in excellent agreement with those of Nied (1981) for $a/b > 0.10$ while their results also agreed with the asymptotic series solution given by Selvadurai and Singh (1985) for small a/b. Hence the results of Clements and Ang are taken for comparison purposes. From Table 2 it can be seen that the present numerical results are in excellent agreement with those of Clements and Ang over the entire range $0 < a/b < 1$. These results were obtained by taking 20 terms in each series.

The terms of the series are given in eqns (74)–(77) and for uniform loading some of these integrals may be analytically evaluated. A very accurate approximate expression for the stress intensity factors of a uniformly loaded annular crack in a homogeneous material can be obtained as follows. Substituting $m = 0$, $\alpha = 0$ and $Q(r_0) = 1$ into eqns (74) and (75), the integrals can be evaluated analytically giving

$$\sigma_{a,0}(r) = \pi\sigma_0\left[\frac{2\sqrt{b^2-a^2}}{\sqrt{a^2-r^2}} - \frac{\pi}{2} - \sin^{-1}\left(\frac{b^2+r^2-2a^2}{b^2-r^2}\right)\right], \tag{80}$$

$$\sigma_{b,0}(r) = \pi\sigma_0\left[\frac{2\sqrt{b^2-a^2}}{\sqrt{r^2-b^2}} - \frac{\pi}{2} - \sin^{-1}\left(\frac{2b^2-a^2-r^2}{r^2-a^2}\right)\right]. \tag{81}$$

Inserting the above expressions into eqns (76) and (77) with $n = 1$ gives the terms $\sigma_{a,1}(r)$ and $\sigma_{b,1}(r)$ in integral form. These integrals can be evaluated for $r = a$ and $r = b$ as in the previous case of shear loading leading to the following two term approximations to the stress intensity factors

$$K_{\rm I}^a = \frac{4}{\pi^2}\sigma_0\sqrt{a}\sqrt{\varepsilon^{-2}-1}, \tag{82}$$

$$K_{\rm I}^b = \frac{4}{\pi^2}\sigma_0\sqrt{b}\tan^{-1}(\sqrt{\varepsilon^{-2}-1}), \tag{83}$$

where $\varepsilon = a/b$. The above expressions are accurate for small ε. For small ε the stress intensity factors behave as $K_{\rm I}^a = 4\sigma_0\sqrt{b}/(\pi^2\sqrt{\varepsilon})$ and $K_{\rm I}^b = 2\sigma_0\sqrt{b}/\pi$ which agree with the first term of the asymptotic series derived by Selvadurai and Singh (1985). Adding additional terms to eqns (82) and (83) leads to the following approximate expressions:

Table 2. Comparison of stress intensity factors for a uniformly loaded annular crack in an infinite homogeneous material with previous results and approximate expressions

			Clements and Ang		Approximate expressions	
a/b	$\bar{K}_{\rm I}^a$	$\bar{K}_{\rm I}^b$	$\bar{K}_{\rm I}^a$	$\bar{K}_{\rm I}^b$	$\bar{K}_{\rm I}^a$	$\bar{K}_{\rm I}^b$
0.01	4.090	0.634	4.069	0.634	4.067	0.634
0.1	1.323	0.610	1.323	0.610	1.318	0.609
0.2	0.950	0.581	0.950	0.581	0.945	0.580
0.3	0.775	0.548	0.775	0.548	0.771	0.549
0.4	0.659	0.513	0.659	0.513	0.657	0.514
0.5	0.568	0.473	0.569	0.473	0.567	0.474
0.6	0.488	0.427	0.488	0.428	0.488	0.430
0.7	0.410	0.374	0.410	0.374	0.411	0.377
0.8	0.327	0.309	0.327	0.309	0.328	0.311
0.9	0.227	0.221	0.227	0.221	0.229	0.223
0.99	0.071	0.071	0.071	0.071	0.072	0.071

$$K_I^a = \frac{4}{\pi^2}\sigma_0\sqrt{b}\sqrt{\varepsilon}\left(\sqrt{\varepsilon^{-2}-1}+\frac{\sqrt{2}}{40}\pi^2\sqrt{1-\varepsilon}\right), \tag{84}$$

$$K_I^b = \frac{4}{\pi^2}\sigma_0\sqrt{b}\left[\tan^{-1}(\sqrt{\varepsilon^{-2}-1})+\frac{\sqrt{2}}{40}\pi^2\varepsilon\sqrt{1-\varepsilon}\right]. \tag{85}$$

The additional terms were deduced by comparing with the numerical data. Table 2 also gives the numerical results from the approximate expressions above. The maximum error of 0.5% occurs in $\bar{K}_I^a$ at a value of $a/b = 0.2$. The approximate expressions provide a simple means of obtaining very accurate values with little computational effort.

Now consider the inhomogeneous case for uniform pressure and general values of α. The non-dimensionalized stress intensity factors $\bar{K}_I^a$ and $\bar{K}_I^b$ are defined as $K_I = \sigma_0 b^{(1-\alpha)/2}\bar{K}_I$ and they are given in Table 3 for various values of the ratio a/b and the parameter α. Table 3(a) gives the results for $0 \leqslant \alpha < 1$ and $0 < a/b \leqslant 0.5$ while Table 3(b) is for $0 \leqslant \alpha < 1$ and $0.5 < a/b < 1.0$.

First values of the stress intensity factor at the inner radius are examined. Table 3(a) gives the stress intensity factor $\bar{K}_I^a$ $(0 < a/b \leqslant 0.5)$ and for a given value of a/b it generally increases with α to a maximum and then decreases to a limiting value of approximately 1.0 for $\alpha \to 1$. The behavior for $a/b = 0.01$ is strictly decreasing since the peak value in this case occurs at $\alpha = 0$. The location of the peak shifts as the ratio a/b is increased. As a/b increases, the peak occurs at increasing values of α. For $a/b = 0.01, 0.1, 0.2, 0.3, 0.4, 0.5$ the peak is located at $\alpha = 0.0, 0.3, 0.5, 0.7, 0.8, 0.9$ respectively. The magnitude of this maximum value decreases with increasing a/b. From Table 3(b) it is evident that no peak occurs in this range of a/b. In this region $(0.6 < a/b < 1.0)$ the values increase monotonically from a minimum at $\alpha = 0$ to approximately 1.0 at $\alpha = 1.0$.

Attention is now focused on the variation of $\bar{K}_I^a$ with a/b for a fixed value of α. In this case the stress intensity factor decreases uniformly as a/b is increased. As $a/b \to 0$, the values of $\bar{K}_I^a$ becomes infinite but the strength of the singularity decreases as α is increased. On the other hand, as $a/b \to 1$ $\bar{K}_I^a$ tends to zero as indicated in the numerical results and eqn (86) below.

Table 3(a). Non-dimensionalized stress intensity factors for a uniformly loaded annular crack in an inhomogeneous material, $0 \leqslant \alpha < 1$, $0 < a/b \leqslant 0.5$

a/b		0.01	0.1	0.2	0.3	0.4	0.5
$\alpha = 0$	$\bar{K}_I^a$	4.090	1.323	0.950	0.775	0.659	0.568
	$\bar{K}_I^b$	0.634	0.610	0.580	0.548	0.512	0.473
$\alpha = 0.1$	$\bar{K}_I^a$	3.903	1.406	1.036	0.858	0.739	0.645
	$\bar{K}_I^b$	0.697	0.676	0.648	0.617	0.581	0.541
$\alpha = 0.2$	$\bar{K}_I^a$	3.633	1.457	1.106	0.931	0.813	0.718
	$\bar{K}_I^b$	0.756	0.738	0.713	0.684	0.649	0.610
$\alpha = 0.3$	$\bar{K}_I^a$	3.305	1.481	1.159	0.994	0.881	0.788
	$\bar{K}_I^b$	0.809	0.796	0.774	0.747	0.715	0.677
$\alpha = 0.4$	$\bar{K}_I^a$	2.945	1.476	1.193	1.045	0.940	0.853
	$\bar{K}_I^b$	0.858	0.848	0.829	0.806	0.777	0.742
$\alpha = 0.5$	$\bar{K}_I^a$	2.572	1.445	1.209	1.080	0.988	0.910
	$\bar{K}_I^b$	0.900	0.892	0.878	0.859	0.834	0.804
$\alpha = 0.6$	$\bar{K}_I^a$	2.206	1.390	1.203	1.099	1.022	0.957
	$\bar{K}_I^b$	0.935	0.930	0.920	0.905	0.885	0.860
$\alpha = 0.7$	$\bar{K}_I^a$	1.858	1.313	1.179	1.101	1.042	0.992
	$\bar{K}_I^b$	0.963	0.960	0.953	0.942	0.928	0.909
$\alpha = 0.8$	$\bar{K}_I^a$	1.538	1.221	1.135	1.085	1.045	1.011
	$\bar{K}_I^b$	0.984	0.982	0.978	0.971	0.961	0.949
$\alpha = 0.9$	$\bar{K}_I^a$	1.253	1.116	1.076	1.051	1.032	1.015
	$\bar{K}_I^b$	0.996	0.995	0.993	0.990	0.986	0.980
$\alpha = 0.99$	$\bar{K}_I^a$	1.027	1.015	1.011	1.009	1.007	1.005
	$\bar{K}_I^b$	1.000	1.000	1.000	1.000	0.999	0.999

$$\begin{bmatrix} K_I^a \\ K_I^b \end{bmatrix} = \sigma_0 b^{(1-\alpha)/2}\begin{bmatrix} \bar{K}_I^a \\ \bar{K}_I^b \end{bmatrix}$$

Table 3(b). Non-dimensionalized stress intensity factors for a uniformly loaded annular crack in an inhomogeneous material, $0 \leq \alpha < 1$, $0.5 < a/b < 1.0$

a/b		0.6	0.7	0.8	0.9	0.99
$\alpha = 0$	$\bar{K}_I^a$	0.490	0.411	0.327	0.227	0.071
	$\bar{K}_I^b$	0.427	0.374	0.309	0.221	0.071
$\alpha = 0.1$	$\bar{K}_I^a$	0.560	0.478	0.389	0.280	0.098
	$\bar{K}_I^b$	0.495	0.439	0.379	0.274	0.098
$\alpha = 0.2$	$\bar{K}_I^a$	0.632	0.548	0.456	0.340	0.134
	$\bar{K}_I^b$	0.563	0.507	0.435	0.333	0.134
$\alpha = 0.3$	$\bar{K}_I^a$	0.704	0.620	0.527	0.407	0.180
	$\bar{K}_I^b$	0.632	0.577	0.505	0.400	0.180
$\alpha = 0.4$	$\bar{K}_I^a$	0.773	0.693	0.602	0.482	0.239
	$\bar{K}_I^b$	0.701	0.648	0.579	0.474	0.239
$\alpha = 0.5$	$\bar{K}_I^a$	0.838	0.764	0.679	0.564	0.314
	$\bar{K}_I^b$	0.767	0.719	0.655	0.554	0.313
$\alpha = 0.6$	$\bar{K}_I^a$	0.895	0.830	0.755	0.650	0.406
	$\bar{K}_I^b$	0.828	0.788	0.731	0.640	0.405
$\alpha = 0.7$	$\bar{K}_I^a$	0.942	0.890	0.828	0.740	0.519
	$\bar{K}_I^b$	0.885	0.852	0.806	0.730	0.518
$\alpha = 0.8$	$\bar{K}_I^a$	0.977	0.940	0.896	0.831	0.655
	$\bar{K}_I^b$	0.932	0.910	0.878	0.822	0.653
$\alpha = 0.9$	$\bar{K}_I^a$	0.998	0.979	0.955	0.919	0.815
	$\bar{K}_I^b$	0.972	0.960	0.943	0.913	0.813
$\alpha = 0.99$	$\bar{K}_I^a$	1.003	1.001	0.998	0.994	0.982
	$\bar{K}_I^b$	0.998	0.997	0.995	0.992	0.980

$$\begin{bmatrix} K_I^a \\ K_I^b \end{bmatrix} = \sigma_0 b^{(1-\alpha)/2} \begin{bmatrix} \bar{K}_I^a \\ \bar{K}_I^b \end{bmatrix}$$

Examining the stress intensity factor $\bar{K}_I^b$ at the outer radius, a strictly monotonic behavior is observed for a fixed a/b ratio. This stress intensity factor increases from a minimum at $\alpha = 0$ to a maximum of near 1.0 at $\alpha = 1.0$. This behavior holds for all values of a/b. For a fixed value of α, $\bar{K}_I^b$ decreases as a/b is increased and tends to zero for $a/b \to 1$ as eqn (86) below shows.

The results in Table 3(a, b) for a uniformly loaded annular crack in an inhomogeneous material appear to be new. The results can be partially checked by verifying some limiting cases. The results for $\alpha = 0$ represent a homogeneous material and this limiting case has been discussed previously. The limit $a/b \to 1$ corresponds to the two-dimensional problem of a crack with a length of $(b-a)$ in an inhomogeneous material under a uniform pressure σ_0. The stress intensity factor for this problem has been given by Sih and Chen (1981) as

$$K_I^a = K_I^b = \sigma_0 b^{(1-\alpha)/2} \sqrt{\pi} \left(\tfrac{1}{2}\right)^{(1-\alpha)/2} \frac{(1-a/b)^{(1-\alpha)/2}}{\Gamma\left(1-\frac{\alpha}{2}\right)\Gamma\left(\frac{1+\alpha}{2}\right)}, \tag{86}$$

where Γ is the gamma function. For $a/b = 0.99$, the present numerical results in Table 3(b) are in excellent agreement with the above formula. For $a/b \to 0$, the stress intensity factor at the outer radius approaches that for a uniformly loaded penny-shaped crack. This limit is given in Kassir and Sih (1975, pp. 382–392) as

$$K_I^b = \sigma_0 b^{(1-\alpha)/2} \frac{2}{\pi} \cos\left(\frac{\pi\alpha}{2}\right) \frac{1}{(1-\alpha)}. \tag{87}$$

The results for $\bar{K}_I^b$ in Table 3(a) for $a/b = 0.01$ are also in excellent agreement with the above expression.

The second loading condition corresponds to bending and is obtained by taking $m = 1$ and $Q(r) = r/b$. The non-dimensionalized stress intensity factors $\bar{K}_I^a$ and $\bar{K}_I^b$ are now defined as $K_{\mathrm{I}} = \sigma_0 \cos(\theta) b^{(1-\alpha)/2} \bar{K}_{\mathrm{I}}$. Numerical results are given in Table 4(a) for $0 \leqslant \alpha < 1$ and $0 < a/b \leqslant 0.5$ while Table 4(b) is for $0 \leqslant \alpha < 1$ and $0.5 < a/b < 1.0$.

Again considering $\bar{K}_{\mathrm{I}}^a$ first in Table 4(a), the variation with α for a fixed a/b is observed to be analogous with the previous case. A peak occurs in the stress intensity factor as α is increased from zero to one. Contrary to the previous case, this peak value now increases with increasing a/b. From Table 4(b) the behavior in this range of a/b is similar to the previous loading.

Table 4(a). Non-dimensionalized stress intensity factors for an annular crack in an inhomogeneous material under bending, $0 \leqslant \alpha < 1$, $0 < a/b \leqslant 0.5$

a/b		0.01	0.1	0.2	0.3	0.4	0.5
$\alpha = 0$	$\bar{K}_{\mathrm{I}}^a$	0.054	0.170	0.236	0.280	0.309	0.324
	$\bar{K}_{\mathrm{I}}^b$	0.424	0.423	0.422	0.418	0.410	0.395
$\alpha = 0.1$	$\bar{K}_{\mathrm{I}}^a$	0.048	0.172	0.247	0.301	0.338	0.360
	$\bar{K}_{\mathrm{I}}^b$	0.482	0.482	0.480	0.476	0.468	0.454
$\alpha = 0.2$	$\bar{K}_{\mathrm{I}}^a$	0.043	0.171	0.255	0.317	0.363	0.395
	$\bar{K}_{\mathrm{I}}^b$	0.541	0.541	0.539	0.536	0.529	0.516
$\alpha = 0.3$	$\bar{K}_{\mathrm{I}}^a$	0.037	0.167	0.259	0.330	0.385	0.426
	$\bar{K}_{\mathrm{I}}^b$	0.600	0.600	0.599	0.596	0.590	0.579
$\alpha = 0.4$	$\bar{K}_{\mathrm{I}}^a$	0.032	0.161	0.259	0.338	0.402	0.453
	$\bar{K}_{\mathrm{I}}^b$	0.660	0.660	0.659	0.657	0.652	0.642
$\alpha = 0.5$	$\bar{K}_{\mathrm{I}}^a$	0.027	0.153	0.256	0.342	0.415	0.476
	$\bar{K}_{\mathrm{I}}^b$	0.720	0.720	0.720	0.718	0.714	0.706
$\alpha = 0.6$	$\bar{K}_{\mathrm{I}}^a$	0.022	0.144	0.250	0.341	0.422	0.493
	$\bar{K}_{\mathrm{I}}^b$	0.780	0.780	0.779	0.778	0.775	0.769
$\alpha = 0.7$	$\bar{K}_{\mathrm{I}}^a$	0.019	0.134	0.240	0.337	0.425	0.504
	$\bar{K}_{\mathrm{I}}^b$	0.838	0.838	0.837	0.837	0.835	0.830
$\alpha = 0.8$	$\bar{K}_{\mathrm{I}}^a$	0.015	0.123	0.229	0.328	0.422	0.510
	$\bar{K}_{\mathrm{I}}^b$	0.894	0.894	0.894	0.893	0.892	0.889
$\alpha = 0.9$	$\bar{K}_{\mathrm{I}}^a$	0.012	0.111	0.215	0.316	0.414	0.508
	$\bar{K}_{\mathrm{I}}^b$	0.949	0.949	0.948	0.948	0.948	0.946
$\alpha = 0.99$	$\bar{K}_{\mathrm{I}}^a$	0.010	0.101	0.202	0.302	0.402	0.502
	$\bar{K}_{\mathrm{I}}^b$	0.995	0.995	0.995	0.995	0.995	0.995

$$\begin{bmatrix} K_{\mathrm{I}}^a \\ K_{\mathrm{I}}^b \end{bmatrix} = \sigma_0 \cos\theta b^{(1-\alpha)/2} \begin{bmatrix} \bar{K}_{\mathrm{I}}^a \\ \bar{K}_{\mathrm{I}}^b \end{bmatrix}$$

Table 4(b). Non-dimensionalized stress intensity factors for an annular crack in an inhomogeneous material under bending, $0 \leqslant \alpha < 1$, $0.5 < a/b < 1.0$

a/b		0.6	0.7	0.8	0.9	0.99
$\alpha = 0$	$\bar{K}_{\mathrm{I}}^a$	0.325	0.311	0.278	0.212	0.071
	$\bar{K}_{\mathrm{I}}^b$	0.372	0.339	0.290	0.214	0.070
$\alpha = 0.1$	$\bar{K}_{\mathrm{I}}^a$	0.368	0.359	0.328	0.260	0.098
	$\bar{K}_{\mathrm{I}}^b$	0.432	0.398	0.347	0.265	0.097
$\alpha = 0.2$	$\bar{K}_{\mathrm{I}}^a$	0.411	0.409	0.382	0.315	0.133
	$\bar{K}_{\mathrm{I}}^b$	0.495	0.462	0.410	0.323	0.133
$\alpha = 0.3$	$\bar{K}_{\mathrm{I}}^a$	0.451	0.458	0.439	0.376	0.179
	$\bar{K}_{\mathrm{I}}^b$	0.559	0.527	0.477	0.389	0.179
$\alpha = 0.4$	$\bar{K}_{\mathrm{I}}^a$	0.489	0.506	0.497	0.442	0.238
	$\bar{K}_{\mathrm{I}}^b$	0.625	0.596	0.549	0.462	0.238
$\alpha = 0.5$	$\bar{K}_{\mathrm{I}}^a$	0.522	0.552	0.556	0.515	0.311
	$\bar{K}_{\mathrm{I}}^b$	0.691	0.666	0.623	0.541	0.312
$\alpha = 0.6$	$\bar{K}_{\mathrm{I}}^a$	0.551	0.594	0.614	0.591	0.403
	$\bar{K}_{\mathrm{I}}^b$	0.757	0.736	0.700	0.627	0.404
$\alpha = 0.7$	$\bar{K}_{\mathrm{I}}^a$	0.574	0.631	0.670	0.671	0.514
	$\bar{K}_{\mathrm{I}}^b$	0.821	0.806	0.777	0.717	0.517
$\alpha = 0.8$	$\bar{K}_{\mathrm{I}}^a$	0.591	0.662	0.720	0.750	0.649
	$\bar{K}_{\mathrm{I}}^b$	0.884	0.874	0.854	0.811	0.652
$\alpha = 0.9$	$\bar{K}_{\mathrm{I}}^a$	0.599	0.686	0.765	0.828	0.807
	$\bar{K}_{\mathrm{I}}^b$	0.944	0.939	0.929	0.906	0.813
$\alpha = 0.99$	$\bar{K}_{\mathrm{I}}^a$	0.602	0.701	0.798	0.895	0.973
	$\bar{K}_{\mathrm{I}}^b$	0.994	0.994	0.993	0.991	0.980

$$\begin{bmatrix} K_{\mathrm{I}}^a \\ K_{\mathrm{I}}^b \end{bmatrix} = \sigma_0 \cos\theta b^{(1-\alpha)/2} \begin{bmatrix} \bar{K}_{\mathrm{I}}^a \\ \bar{K}_{\mathrm{I}}^b \end{bmatrix}$$

Focusing on the variation of $\bar{K}_I^a$ with a/b for a fixed value of α it is apparent that a peak also exists in this direction. The values of $\bar{K}_I^a$ tend to zero for $a/b \to 0$ and $a/b \to 1$ while attaining an intermediate maximum value.

The stress intensity factor $\bar{K}_I^b$ exhibits the same variation with α as the uniform loading case. That is, for all values of a/b it has a monotonic increase from a minimum value at $\alpha = 0$ to a maximum of approximately 1.0 at $\alpha = 1.0$. On the other hand, for a fixed value of α it decreases monotonically as a/b is increased.

The numerical results can be partially verified by studying some limiting cases. The top two rows in Table 4(a, b) correspond to a homogeneous material. This case was analyzed by Nied and Erdogan (1983). They did not give tabulated numerical results; however the present numbers agree very well with their figures once the different definition of stress intensity factor is accounted for. In the limit $a/b \to 1$, a two-dimensional plane strain crack of length $(b-a)$ under uniform loading of magnitude $\sigma_0 \cos(\theta)$ is obtained. Hence the numerical results in the last column of Table 4(b) should be very close to the last column of Table 3(b) (and equivalently eqn (86) with the term $\cos(\theta)$ added to the right hand side). Comparison of the last column of these two tables for $a/b = 0.99$ reveals almost identical numbers. The slight difference results from the $\cos(\theta)$ stress variation which still has a small effect for $a/b = 0.99$. Finally the results for $\bar{K}_I^b$ as $a/b \to 0$ may be checked against the analytical solution for a penny-shaped crack. Using the crack face loading in eqn (71) with $m = 1$ and $Q(r) = r/b$, the stress intensity factor can be analytically evaluated by integrating the Green's function in eqn (34) giving

$$K_I^b = \sigma_0 \cos(\theta) b^{(1-\alpha)/2} \frac{2}{\pi} \cos\left(\frac{\pi\alpha}{2}\right) \left\{ \frac{1}{(1-\alpha)} - \frac{1}{(3-\alpha)} \right\}. \tag{88}$$

The results for $\bar{K}_I^b$ in the first column of Table 4(a) for $a/b = 0.01$ agree very well with the above limiting case.

5. DISCUSSION

This paper has developed a basic solution procedure for certain mixed boundary value problems in potential theory. The solution here is appropriate for boundary value problems with annular type circular or non-circular regions. The method converts a pair of coupled integral equations to two non-coupled equations and writes their solution in series form. The terms of the series are evaluated as successive numerical integrations of elementary functions and hence the method is mathematically simple and tractable. Furthermore, it is illustrated that the method can handle certain types of inhomogeneous medium as easily as the homogeneous case. As an example, the solution procedure was applied to fracture mechanics and numerical results were given for a circular annular crack in an inhomogeneous material where the modulus varies as a power law from the crack plane $z = 0$ ($E = E_0|z|^\alpha$, E_0 = constant, $0 < \alpha < 1$). The results were compared to previous calculations and limiting cases available and the numerical calculations were found to be of very high accuracy.

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